GLUON REGGE TRAJECTORY IN THE TWO-LOOP APPROXIMATION [*]

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Abstract

Evaluating two-loop integrals in the transverse momentum space we obtain the trajectory of the Reggeized gluon in QCD in an explicit form in the two-loop approximation. It is presented as an expansion in powers of $(D-4)$ for the space-time dimension D tending to the physical value $D = 4$. As the result of a remarkable cancellation the third order pole in $D-4$ disappears in the trajectory and the expansion starts with $(D-4)^{-2}$.

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1. INTRODUCTION

Dynamics of semihard processes (i.e. processes with the c.m.s. energy $\sqrt{s}$ much larger than a typical virtuality Q) is one of the most interesting problems of modern high energy physics. There are a lot of papers devoted to the theoretical description of the processes (see, for example, Refs. [1,2] and references therein). Neverthe-less, the present situation here is not quite satisfactory, even in the framework of perturbative QCD which, presumably, can be applied to such processes [1]. The BFKL equation [3] used for the calculation of the small $x = \frac{Q^2}{s}$ behaviour of cross-sections is responsible for the leading $\ln(1/x)$ terms only. To define a region of its applicability as well as to fix a scale of virtualities of the running coupling constant $\alpha_s(Q^2)$ in the equation one has to know radiative corrections to the kernel of the equation. The crucial point in the programme of calculation of the next-to-leading corrections [4] is the gluon Reggeization in QCD. The Reggeization formed a basis of the original derivation of the BFKL equation [4] and was proved in the leading $\ln(1/x)$ approximation for elastic and inelastic processes [5]. In this approximation the gluon trajectory

$$j(t) = 1 + \omega(t) \ ,$$

$$\omega(t) = \omega^{(1)}(t) + \omega^{(2)}(t) + \cdots \tag{1}$$

can be written as an integral in the transverse momentum space:

$$\omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{(D-1)}} \frac{N}{2} \int \frac{d^{D-2}k}{\vec{k}^2(\vec{q}-\vec{k})^2} \ , \qquad t = -\vec{q}^{\,2} \ , \tag{2}$$

corresponding to a simple Feynman diagram of $(D-2)$-dimensional field theory. Here and below N is the number of colours ($N = 3$ for QCD). The integral is divergent in the physical case $D = 4$. In fact one gets

$$\omega^{(1)}(t) = -\bar{g}^2 (\vec{q}^{\,2})^\epsilon \frac{2}{\epsilon} \frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)} \ , \tag{3}$$

where

$$\epsilon = \frac{D}{2} - 2 \ , \qquad \bar{g}^2 = \frac{g^2 N}{(4\pi)^{\frac{D}{2}}} \Gamma(1-\epsilon) \ . \tag{4}$$

This divergency is effectively cancelled in the kernel of the BFKL equation

$$\mathcal{K}(\vec{q}, \vec{q}\,') = 2\omega(t)\delta(\vec{q} - \vec{q}\,') + \mathcal{K}_{real}(\vec{q}, \vec{q}\,') \,, \tag{5}$$

$$\mathcal{K}_{real}^{(1)}(\vec{q}, \vec{q}\,') = \frac{\bar{g}^2}{\pi^{1+\epsilon}\Gamma(1+\epsilon)} \frac{4}{(\vec{q} - \vec{q}\,')^2} \,, \tag{6}$$

which is not singular at $D = 4$ for integration with any smooth function of $\vec{q}\,'$.

In the next-to-leading approximation Reggeization was checked and the correction to the trajectory was presented in the form of two-loop integrals in the transverse momentum space [6,7]. However, it appears that we have to calculate the integrals because of increased infrared divergency of separate terms in the correction to the kernel which diverge as $(D - 4)^{-3}$. So, contrary to what happened in the leading order, where the divergency cancellation was organized by a suitable rearrangement of virtual and real contributions to the kernel without going to $D \neq 4$, in the correction one hardly can hope to do it. Therefore, we need to calculate the correction to the trajectory in an explicit form. In the present paper we solve this problem.

2. REPRESENTATION OF THE QUARK AND GLUON CORRECTIONS

The two-loop correction contains quark as well as gluon contributions.

Let us first consider the quark contribution. Of course, here the case of light quarks is the most important one. Due to the cancellation of infrared terms in the equation masses of the light quarks should not affect the BFKL dynamics, so we can put them equal to zero. Therefore, according to Refs. [6,7] this contribution can be written as

$$\omega_q^{(2)}(t) = \frac{\bar{g}^4 \vec{q}^{\,2}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \frac{4n_f}{N\epsilon} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q} - \vec{q}_1)^2} \int_0^1 dx (x(1-x))^{1+\epsilon} \left((\vec{q}^{\,2})^\epsilon - 2(\vec{q}_1^{\,2})^\epsilon\right) \,. \tag{7}$$

where n_f is the number of light flavours. Making use of the generalized Feynman parametrization

$$\prod_{i=1}^{n} a_i^{-\alpha_i} = \frac{\Gamma\left(\sum_{i=1}^{n} \alpha_i\right)}{\prod_{i=1}^{n} \Gamma(\alpha_i)} \left(\prod_{i=1}^{n} \int_0^1 dx_i x_i^{\alpha_i-1}\right) \frac{\delta(1 - \sum_{i=1}^{n} x_i)}{\left(\sum_{i=1}^{n} a_i x_i\right)^{\sum_{i=1}^{n} \alpha_i}} \tag{8}$$

the integration in Eq. (7) can be performed without any difficulty and leads to

$$\omega_q^{(2)}(t) = \bar{g}^4 (\vec{q}^{\,2})^{2\epsilon} \frac{4 n_f}{N\epsilon} \frac{\Gamma^2(2+\epsilon)}{\Gamma(4+2\epsilon)} \left[\frac{2}{\epsilon} \frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)} - \frac{3}{\epsilon} \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \frac{\Gamma(1+\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1+3\epsilon)}\right] . \tag{9}$$

For the case $\epsilon \to 0$ Eq. (9) gives

$$\omega_q^{(2)}(t) = -\bar{g}^4 (\vec{q}^{\,2})^{2\epsilon} \frac{2 n_f}{3N} \frac{1}{\epsilon^2} \left[1 - \frac{5}{3}\epsilon - \left(\frac{\pi^2}{3} - \frac{28}{9}\right)\epsilon^2\right] . \tag{10}$$

The gluon contribution is more complicated. It reads [7]

$$\omega_g^{(2)}(t) = (\vec{q}^{\,2})^{2\epsilon} \left\{ [\psi(1) + 2\psi(\epsilon) - \psi(1-\epsilon) - 2\psi(1+2\epsilon) \right.$$

$$\left. + \frac{1}{(1+2\epsilon)} \left(\frac{1}{\epsilon} + \frac{1+\epsilon}{2(3+2\epsilon)}\right)\right] (I_1 - 2J_1) - 2J_2 + 2J_3 - I_2 \right\} , \tag{11}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function, $\psi(z) = d\ln\Gamma(z)/dz$, and the quantities I_i, J_i are two-loop integrals in the transverse momentum space. They can be written in the following form:

$$I_i = \left(\frac{g^2 N}{2(2\pi)^{D-1}}\right)^2 \int \frac{d^{(D-2)} q_1 d^{(D-2)} q_2 (\vec{q}^{\,2})^{6-D}}{\vec{q}_1^{\,2} (\vec{q}_1 - \vec{q})^2 \vec{q}_2^{\,2} (\vec{q}_2 - \vec{q})^2} a_i , \tag{12}$$

with

$$a_1 = 1 , \qquad a_2 = \ln\left(\frac{(\vec{q}_1 - \vec{q}_2)^2}{\vec{q}^{\,2}}\right) \tag{13}$$

and

$$J_i = \left(\frac{g^2 N}{2(2\pi)^{D-1}}\right)^2 \int \frac{d^{(D-2)} q_1 d^{(D-2)} q_2 (\vec{q}^{\,2})^{5-D}}{\vec{q}_1^{\,2} \vec{q}_2^{\,2} (\vec{q} - \vec{q}_1 - \vec{q}_2)^2} b_i , \tag{14}$$

with

$$b_1 = 1 \; , \qquad b_2 = \ln\left(\frac{(\vec{q} - \vec{q}_2)^2}{\vec{q}^{\,2}}\right) \; , \qquad b_3 = \ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right) \; . \qquad (15)$$

Let us note that because of the logarithmic factors (coming from the integration over longitudinal momentum components) the integrals I_i and J_i, with $i \geq 2$, do not correspond to usual Feynman diagrams of $D - 2$-dimensional field theory.

The integral I_1 is given by the square of the integral for the leading contribution to the trajectory (2) and reads

$$I_1 = \bar{g}^4 \left(\frac{2}{\epsilon} \frac{\Gamma^2(1 + \epsilon)}{\Gamma(1 + 2\epsilon)}\right)^2 \; . \qquad (16)$$

The calculation of the integral J_1 is not much more complicated than that of I_1. It can be performed by the subsequent integration over $\vec{q}_1$ (using the Feynman parametrization to join the denominators $\vec{q}_1^{\,2}$ and $(\vec{q} - \vec{q}_1 - \vec{q}_2)^2$) and $\vec{q}_2$. At the second step a non-integer power of $(\vec{q}_2 - \vec{q})^2$ appears in the denominator, so that we need to use the generalized parametrization (8). One easily obtains

$$J_1 = \bar{g}^4 \frac{3}{\epsilon^2} \frac{\Gamma(1 - 2\epsilon)\Gamma^3(1 + \epsilon)}{\Gamma^2(1 - \epsilon)\Gamma(1 + 3\epsilon)} \; . \qquad (17)$$

The integrals J_2 and J_3 can be calculated in the same way with the help of the representation

$$\ln a = -\frac{d}{d\nu} a^{-\nu}|_{\nu=0} \; . \qquad (18)$$

The result is

$$J_2 = J_1[\psi(1 - \epsilon) + \psi(2\epsilon) - \psi(1 - 2\epsilon) - \psi(3\epsilon)] \; , \qquad (19)$$

$$J_3 = J_1[\psi(1) + \psi(\epsilon) - \psi(1 - 2\epsilon) - \psi(3\epsilon)] \; . \qquad (20)$$

The calculation of the integral I_2 is much more complicated and it appears that in the general case $D \neq 4$ it can be given only in terms of infinite series. [1] Instead

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[1] We are thankful to A.V. Kotikov who paid our attention on papers [8] where I_2 is expressed in terms of generalized hypergeometric series

in the interesting for physical applications case $D \to 4$, the integral is expressed in terms of the well-known values of the Riemann zeta function. Because of the importance of its contribution we present the calculation of the integral in the following section. The result is

$$I_2 = \frac{\bar{g}^4}{\epsilon^2} \left[-\frac{1}{\epsilon} + 2\epsilon \psi'(1) - 13\epsilon^2 \psi''(1) \right] . \tag{21}$$

Let us remind that

$$\psi'(1) = \zeta(2) = \frac{\pi^2}{6} , \qquad \psi''(1) = -2\zeta(3) \approx -2.404 . \tag{22}$$

Expanding for the physical case $\epsilon \to 0$ the integrals I_1 and J_1-J_3, given respectively in Eqs. (16), (17), (19) and (20), and the coefficient of $I_1 - 2J_1$ in Eq. (11) we find

$$I_1 = \frac{4\bar{g}^4}{\epsilon^2} [1 - 2\epsilon^2 \psi'(1) - 2\epsilon^3 \psi''(1)] , \tag{23}$$

$$J_1 = \frac{3\bar{g}^4}{\epsilon^2} \left[1 - 2\epsilon^2 \psi'(1) - 5\epsilon^3 \psi''(1) \right] , \tag{24}$$

$$J_2 = \frac{\bar{g}^4}{\epsilon^2} \left[-\frac{1}{2\epsilon} + \epsilon \psi'(1) - \frac{19}{2}\epsilon^2 \psi''(1) \right] , \tag{25}$$

$$J_3 = \frac{\bar{g}^4}{\epsilon^2} \left[-\frac{2}{\epsilon} + 4\epsilon \psi'(1) - 8\epsilon^2 \psi''(1) \right] , \tag{26}$$

and

$$\psi(1) + 2\psi(\epsilon) - \psi(1-\epsilon) - 2\psi(1+2\epsilon) + \frac{1}{(1+2\epsilon)} \left(\frac{1}{\epsilon} + \frac{1+\epsilon}{2(3+2\epsilon)} \right) =$$

$$-\frac{1}{\epsilon} - \frac{11}{6} - \left(\psi'(1) - \frac{67}{18} \right) \epsilon - \left(\frac{7}{2}\psi''(1) + \frac{202}{27} \right) \epsilon^2 . \tag{27}$$

Finally, substituting the results (21), (23)-(27) into Eq.(11), we obtain the gluon contribution to the two-loop correction to the gluon trajectory:

$$\omega_g^{(2)}(t) = \bar{g}^4(\vec{q}^{\,2})^{2\epsilon} \frac{1}{\epsilon^2} \left[\frac{11}{3} + \left(2\psi'(1) - \frac{67}{9} \right) \epsilon + \left(\psi''(1) - \frac{22}{3}\psi'(1) + \frac{404}{27} \right) \epsilon^2 \right] . \tag{28}$$

3. CALCULATION OF THE INTEGRAL I_2

We introduce the Feynman parameters x_1 and x_2 to join the denominators depending on q_1 and q_2 correspondingly in Eq. (12), and the parameter z to join the results of these two parametrizations. Then we integrate over $\vec{q}_2$ keeping the difference $\vec{\Delta} = \vec{q}_1 - \vec{q}_2$ fixed and after applying the relation (18) arrive at

$$I_2 = \frac{\bar{g}^4}{\Gamma^2(1-\epsilon)} \int_0^1 dz \int_0^1 dx_1 \int_0^1 dx_2 \left(-\frac{\partial}{\partial \nu}\right) f(\nu, D)|_{\nu=0} , \qquad (29)$$

where

$$f(\nu, D) = \frac{\Gamma\left(5 - \frac{D}{2}\right)}{\pi^{\frac{D-2}{2}}} (\vec{q}^{\,2})^{(6-D+\nu)} z(1-z) \qquad (30)$$

$$\times \int \frac{d^{D-2}\Delta}{(\vec{\Delta}^2)^{\nu} [z(\vec{\Delta}^2 + x_1\vec{q}^2 - 2x_1\vec{\Delta}\cdot\vec{q}) + (1-z)x_2\vec{q}^2 - (z(\vec{\Delta} - x_1\vec{q}) - (1-z)x_2\vec{q})^2]^{5-\frac{D}{2}}} .$$

The integration over $\vec{\Delta}$ with the help of the generalized Feynman parametrization with parameter x and the subsequent integration over x by parts yield

$$f(\nu, D) = \frac{\Gamma(6-D+\nu)}{\Gamma(1+\nu)} (z(1-z))^{2-\frac{D}{2}+\nu} \int_0^1 dx (1-x)^{\nu} \frac{d}{dx} \frac{x^{\frac{D}{2}-2-\nu}}{(Q(x))^{6-D+\nu}} \qquad (31)$$

with

$$Q(x) = zx_1(1-x_1) + (1-z)x_2(1-x_2) + z(1-z)(1-x)(x_1 - x_2)^2 . \qquad (32)$$

It is convenient to present the derivative of $f(\nu, D)$ with respect to ν in the form

$$-\frac{\partial}{\partial \nu} f(\nu, D)|_{\nu=0} = f_1 + f_2 + f_3 , \qquad (33)$$

where

$$f_1 = \left(\psi(1) + \frac{\partial}{\partial D}\right) f(0, D) , \qquad (34)$$

$$f_2 = -\frac{1}{2}\ln(z(1-z))f(0, D) = -\frac{\Gamma(6-D)}{2}\frac{(z(1-z))^{2-\frac{D}{2}}}{(Q(1))^{6-D}}\ln(z(1-z)) , \qquad (35)$$

$$f_3 = -\Gamma(6-D)(z(1-z))^{2-\frac{D}{2}} \int_0^1 dx \ln(1-x)\frac{d}{dx}\frac{x^{\frac{D}{2}-2}}{(Q(x))^{6-D}} . \qquad (36)$$

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Let us denote the contributions of the functions f_i to the integral I_2 in Eq. (29) as $\mathcal{I}_i$. It is easy to see that

$$\mathcal{I}_1 =$$

$$\frac{\bar{g}^4}{\Gamma^2(1-\epsilon)}\left(\psi(1)+\frac{1}{2}\frac{d}{d\epsilon}\right)\frac{\Gamma^2(1-\epsilon)}{\bar{g}^4}I_1 = I_1[\psi(1)+2\psi(\epsilon)-\psi(1-\epsilon)-2\psi(2\epsilon)]\ . \quad (37)$$

The calculation of $\mathcal{I}_2$ and $\mathcal{I}_3$ is not so simple. We will perform it in the limit $\epsilon \to 0$. The idea underlying the calculation is to extract those regions of the integration range which lead to singular contributions for $\epsilon \to 0$ and to put $\epsilon = 0$ in the remaining region. Considering the integrand in Eq. (12) and our way of the Feynman parametrization it is easy to understand that the singular contributions can give only vicinities of points $x_i = 0$ and $x_i = 1$, with $i = 1, 2$. Taking into account the symmetry of the function $Q(x)$ of Eq. (32) under the simultaneous substitutions $x_1 \leftrightarrow 1 - x_1$ and $x_2 \leftrightarrow 1 - x_2$, we can restrict ourselves to the region $x_i > 0$, $x_1 + x_2 < 1$ taking its contribution twice. It is convenient to introduce new variables defined through

$$x_1 = \lambda t \ , \qquad x_2 = (1-\lambda)t \ . \quad (38)$$

In these variables $Q(x)$ becomes

$$Q(x) = tR(x) \ ,$$

$$R(x) = z\lambda(1-\lambda t) + (1-z)(1-\lambda)(1-(1-\lambda)t) + (1-x)z(1-z)(2\lambda-1)^2 t \ , \quad (39)$$

and we have

$$\mathcal{I}_2 = -\frac{\bar{g}^4}{\Gamma^2(1-\epsilon)}\int_0^1 dz(z(1-z))^{-\epsilon}\ln(z(1-z))g(z) \ , \quad (40)$$

$$\mathcal{I}_3 = -\frac{2\bar{g}^4}{\Gamma^2(1-\epsilon)}\int_0^1 dz(z(1-z))^{-\epsilon}\int_0^1 dx\ln(1-x)g(x,z) \ , \quad (41)$$

where

$$g(z) = \Gamma(2-2\epsilon)\int_0^1 dt\, t^{2\epsilon-1}\int_0^1 d\lambda\frac{1}{(R(1))^{2-2\epsilon}} \ , \quad (42)$$

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$$g(x, z) = \Gamma(2 - 2\epsilon) \int_0^1 dt \, t^{2\epsilon - 1} \int_0^1 d\lambda \frac{d}{dx} \frac{x^\epsilon}{(R(x))^{2 - 2\epsilon}} \; . \tag{43}$$

The singular points are $t = 0$, $\lambda = 0$ and $\lambda = 1$. Therefore, we divide the integration region in four pieces:

$$a) \quad 0 < t < \delta \; , \quad 0 < \lambda < 1 \; ; \qquad b) \quad \delta < t < 1 \; , \quad 0 < \lambda < \delta ;$$

$$c) \quad \delta < t < 1 \; , \quad 0 < 1 - \lambda < \delta \; ; \qquad d) \quad \delta < t < 1 \; , \quad \delta < \lambda < 1 - \delta \; . \tag{44}$$

Here we have introduced an intermediate parameter δ assuming that

$$e^{-\frac{1}{\epsilon}} \ll \delta \ll \epsilon^n \tag{45}$$

for any fixed power n. Due to the first of these inequalities one can put $\epsilon = 0$ in the region d) and due to the second one we can neglect t, λ and $(1 - \lambda)$ in comparison with 1 in the regions a), b) and c) correspondingly. Of course, δ is cancelled in the whole expression for $g(z)$ and $g(x, z)$.

In the region a) we have independently on x

$$R = z\lambda + (1 - z)(1 - \lambda) \tag{46}$$

so that the integration in Eqs. (42) and (43) is quite simple. Performing a suitable expansion in ϵ we find

$$g^{(a)}(z) = \Gamma(1 - 2\epsilon) \left[\frac{\delta^{2\epsilon}}{2\epsilon} \left(z^{2\epsilon - 1} + (1 - z)^{2\epsilon - 1} \right) + \frac{2}{1 - 2z} \ln \left(\frac{z}{1 - z} \right) \right] , \tag{47}$$

$$g^{(a)}(x, z) = \frac{\Gamma(1 - 2\epsilon)}{2} \delta^{2\epsilon} x^{\epsilon - 1} \left(z^{2\epsilon - 1} + (1 - z)^{2\epsilon - 1} \right) . \tag{48}$$

Notice that in $g^{(a)}(z)$ we need to keep terms which were neglected in $g^{(a)}(x, z)$.

In the region b) $R(x)$ takes the form

$$R(x) = \lambda + r(x) \; , \qquad r(x) = (1 - z)[1 - t(1 - z(1 - x))] \tag{49}$$

and considerably simplifies for $x = 1$. It is convenient to start with the integration over λ without deriving with respect to x in $g(x, z)$. After that we can put $\epsilon = 0$ in the contributions containing $\delta + r(x)$. The integration over t in the case of $g(z)$ can be done without problems and gives

$$g^{(b)}(z) = \Gamma(1 - 2\epsilon) \left[(1-z)^{2\epsilon-1} \left(\frac{\Gamma^2(2\epsilon)}{\Gamma(4\epsilon)} - \frac{\delta^{2\epsilon}}{2\epsilon} \right) - \frac{1}{1 - z + \delta} \ln \left(\frac{1 - z + \delta}{\delta^2} \right) \right] .$$
(50)

As for $g(x, z)$, using the approximate relation

$$t^{2\epsilon-1} \frac{d}{dx} \frac{x^\epsilon}{(r(x))^{1-2\epsilon}} = \frac{\epsilon}{x} \frac{t^{-1}}{(1-z)^{1-2\epsilon}(1-tx)} + \frac{x^\epsilon z(1 + 2\epsilon \ln t)}{1 - z(1-x)} \frac{d}{dt} (r(x))^{2\epsilon-1} \quad (51)$$

the integration over t yields

$$g^{(b)}(x, z) = \Gamma(1 - 2\epsilon) \left[\frac{(1-z)^{2\epsilon-1} z x^\epsilon}{(1 - z(1-x))} \left((z(1-x))^{2\epsilon-1} - 1 \right) \right.$$

$$\left. + \frac{\epsilon}{x} (1-z)^{2\epsilon-1} \ln \left(\frac{1 - x}{\delta} \right) + \frac{1}{1 - z + \delta} \frac{\partial}{\partial x} \ln(\delta + z(1-z)(1-x)) \right] . \quad (52)$$

The calculation of $g^{(c)}(z)$ and $g^{(c)}(x, z)$ is immediate. In fact, because of the symmetry of the function $R(x)$ in Eq. (39) under the replacement $z \leftrightarrow 1 - z$ and $\lambda \leftrightarrow 1 - \lambda$, these two contributions can be obtained from Eqs. (50) and (52) correspondingly by the substitution $z \leftrightarrow 1 - z$.

Finally we pass to the region d). After putting $\epsilon = 0$ the integration over t becomes quite straightforward. The subsequent integration over λ is very simple in the case of $g(z)$ and gives

$$g^{(d)}(z) = \frac{1}{z + \delta} \ln \left(\frac{z + \delta}{\delta^2} \right) + \frac{1}{1 - z + \delta} \ln \left(\frac{1 - z + \delta}{\delta^2} \right) + \frac{2}{1 - 2z} \ln \left(\frac{1 - z}{z} \right) . \quad (53)$$

For the case of $g(x, z)$ it is convenient to integrate without taking the derivative with respect to x. After integration over t one can represent the coefficient in the term with the logarithm as

$$(z\lambda + (1-z)(1-\lambda))^{-2} = -\frac{d}{d\lambda} \left(\frac{1 - 2\lambda}{z\lambda + (1-z)(1-\lambda)} \right) \quad (54)$$

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and integrating by parts over λ arrives at

$$g^{(d)}(x,z) = -\frac{\partial}{\partial x}\frac{\ln(\delta + z(1-z)(1-x))}{(z+\delta)(1-z+\delta)} \; . \tag{55}$$

Using the results given in Eqs. (47), (50) and (53) we obtain with the required accuracy

$$g(z) = g^{(a)}(z)+g^{(b)}(z)+g^{(b)}(1-z)+g^{(d)}(z) = \Gamma(1-2\epsilon)\frac{\Gamma^2(2\epsilon)}{\Gamma(4\epsilon)}\left(z^{2\epsilon-1} + (1-z)^{2\epsilon-1}\right) \; . \tag{56}$$

Analogously, from Eqs. (48), (52) and (55) for $g(x,z)$ we have

$$g(x,z) = g^{(a)}(x,z) + g^{(b)}(x,z) + g^{(b)}(x,1-z) + g^{(d)}(x,z) =$$

$$\Gamma(1-2\epsilon)\left((1-z)^{2\epsilon-1}z^{2\epsilon} + (1-z)^{2\epsilon}z^{2\epsilon-1}\right)$$

$$\times \left[x^\epsilon(1-x)^{2\epsilon-1} + \frac{\epsilon}{x}\left(3\ln(1-x) + \frac{1}{2}\ln x\right) + \frac{1}{2x}\right] \; . \tag{57}$$

Substituting expressions (56) and (57) into Eqs. (40) and (41) respectively and performing the integration over x and z we obtain

$$\mathcal{I}_2 = -2\bar{g}^4\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\frac{\Gamma^2(2\epsilon)}{\Gamma(4\epsilon)}\frac{\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(1)}[\psi(\epsilon) + \psi(1-\epsilon) - 2\psi(1)] =$$

$$\frac{2\bar{g}^4}{\epsilon^2}\left[\frac{1}{\epsilon} - 2\epsilon\psi'(1) - 10\epsilon^2\psi''(1)\right] \; , \tag{58}$$

and

$$\mathcal{I}_3 = -4\bar{g}^4\frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\frac{\Gamma(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)}$$

$$\times \left[\frac{\Gamma(2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+3\epsilon)}(\psi(2\epsilon) - \psi(1+3\epsilon)) - \frac{1}{2}\psi'(1) - \frac{13}{4}\epsilon\psi''(1)\right] =$$

$$\frac{\bar{g}^4}{\epsilon^2}\left[\frac{1}{\epsilon} + 2\epsilon\psi'(1) + 13\epsilon^2\psi''(1)\right] \; . \tag{59}$$

Using last two equations and Eq. (37) with I_1 given by Eq. (23) we obtain for $I_2 = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ the result (21). [2]

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[2]A.V. Kotikov informed us that he also obtained the same result from the generalized hypergeometric series [8]

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4. RENORMALIZATION AND DISCUSSION

Up to now we worked with unrenormalized quantities. But renormalization is quite trivial. Since the trajectory itself must not be renormalized, we have only to use the renormalized coupling constant g_μ instead of the bare one g . In the $\overline{MS}$ scheme one has

$$g = g_\mu \mu^{-\epsilon} \left[1 + \left(\frac{11}{3} - \frac{2\,n_f}{3\,N} \right) \frac{\bar{g}_\mu^2}{2\epsilon} \right] \ , \tag{60}$$

where

$$\bar{g}_\mu^2 = \frac{g_\mu^2 N \Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}} \ , \tag{61}$$

and for the gluon trajectory in the two-loop approximation we obtain

$$\omega(t) = -\bar{g}_\mu^2 \left(\frac{\vec{q}^{\,2}}{\mu^2} \right)^\epsilon \frac{2}{\epsilon} \left\{ 1 + \frac{\bar{g}_\mu^2}{\epsilon} \left[\left(\frac{11}{3} - \frac{2\,n_f}{3\,N} \right) \left(1 - \frac{\pi^2}{6} \epsilon^2 \right) - \right. \right.$$

$$\left(\frac{\vec{q}^{\,2}}{\mu^2} \right)^\epsilon \left(\frac{11}{6} + \left(\frac{\pi^2}{6} - \frac{67}{18} \right) \epsilon + \left(\frac{202}{27} - \frac{11\pi^2}{18} - \zeta(3) \right) \epsilon^2 - \right.$$

$$\left. \left. \frac{n_f}{3N} \left(1 - \frac{5}{3}\epsilon + \left(\frac{28}{9} - \frac{\pi^2}{3} \right) \epsilon^2 \right) \right) \right] \right\} \ . \tag{62}$$

The remarkable fact which occurred is the cancellation of the third order poles in ϵ existing in separate contributions to the gluon correction $\omega_g^{(2)}(t)$ (11). Possibly this fact indicates that the representation (11) is not the best one. From the other hand a simpler representation for $\omega_g^{(2)}(t)$ means that there should exist a simple expression for the integral I_2. In any case, the cancellation of the terms with ϵ^{-3} is very important for the absence of infrared divergences in the corrections to the BFKL equation. As the result of this cancellation the gluon and quark contributions to $\omega^{(2)}(t)$ have similar infrared behaviour. Moreover, the coefficient of the leading singularity in ϵ is proportional to the coefficient of the one-loop β function. This means that infrared divergences are strongly correlated with ultraviolet ones. The correlation is unique in the sense that it provides the independence of singular

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contributions to $\omega(t)$ on $\vec{q}^{\,2}$. Indeed, expanding Eq. (62) we have

$$\omega(t) = -\bar{g}_\mu^2 \left(\frac{2}{\epsilon} + 2\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right) - \bar{g}_\mu^4 \left[\left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{N}\right)\left(\frac{1}{\epsilon^2} - \ln^2\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right)\right.$$

$$\left. + \left(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N}\right)\left(\frac{1}{\epsilon} + 2\ln\left(\frac{\vec{q}^{\,2}}{\mu^2}\right)\right) - \frac{404}{27} + 2\zeta(3) + \frac{56}{27}\frac{n_f}{N}\right] \ . \qquad (63)$$

Eq. (63) exhibits explicitly all singularities of the trajectory in the two-loop approximation and gives its finite part in the limit $\epsilon \to 0$. Of course, the singularities should cancel putting all the corrections into the BFKL equation. We hope to show this in a subsequent paper.

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